Exercise 36

An elastic band is hung on a hook and a mass is hung on the lower end of the band. When the mass is pulled downward and then released, it vibrates vertically. The equation of motion is $s = 2\cos t + 3\sin t$, $t \geq 0$, where s is measured in centimeters and t in seconds. (Take the positive direction to be downward.)

(a) Find the velocity and acceleration at time t.

(b) Graph the velocity and acceleration functions.

(c) When does the mass pass through the equilibrium position for the first time?

(d) How far from its equilibrium position does the mass travel?

(e) When is the speed the greatest?

Solution

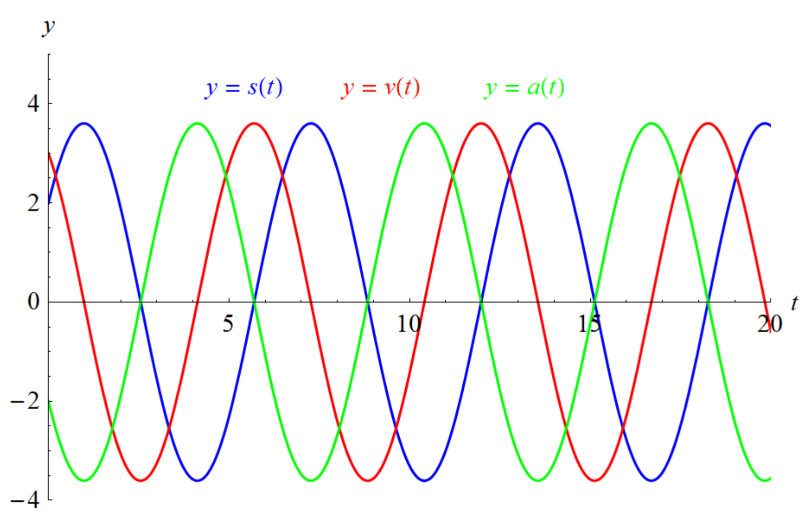
The velocity is the time derivative of the equation of motion.

$$\begin{aligned} v(t) = \frac{ds}{dt} = \frac{d}{dt}(2\cos t + 3\sin t) &= \frac{d}{dt}(2\cos t) + \frac{d}{dt}(3\sin t) \\ &= (-2\sin t) + (3\cos t) \\ &= -2\sin t + 3\cos t \end{aligned}$$

The acceleration is the time derivative of the velocity.

$$\begin{aligned} a(t) = \frac{dv}{dt} = \frac{d}{dt}(-2\sin t + 3\cos t) &= \frac{d}{dt}(-2\sin t) + \frac{d}{dt}(3\cos t) \\ &= (-2\cos t) + (-3\sin t) \\ &= -2\cos t - 3\sin t \end{aligned}$$

A graph showing the position, velocity, and acceleration as a function of time is shown below.



Based on the graph of $y = s(t)$, the mass passes through $s(t) = 0$ for the first time when $t \approx 2.5$ s. The exact time can be found from the equation by setting it equal to zero and solving for t.

$$s(t) = 2\cos t + 3\sin t = 0$$

$$\tan t = -\frac{2}{3}$$

$$t = \tan^{-1}\left(-\frac{2}{3}\right) + \pi \approx 2.55 \text{ s}$$

To find how far the mass goes from equilibrium, it's necessary to write $s(t)$ as a single sinusoidal function with a phase.

$$s(t) = A\cos(t - \phi) = A[\cos t\cos\phi + \sin t\sin\phi] = (A\cos\phi)\cos t + (A\sin\phi)\sin t$$

Comparing this with the given equation of motion $s(t) = 2\cos t + 3\sin t$, we obtain the following system of equations.

$$A\cos\phi = 2$$

$$A\sin\phi = 3$$

Square both sides and add the respective sides to eliminate the phase.

$$A^2\cos^2\phi + A^2\sin^2\phi = 2^2 + 3^2$$

$$A^2 = 13$$

$$A = \sqrt{13} \approx 3.61 \text{ cm}$$

This is how far the mass extends from equilibrium. It's consistent with the graph of $y = s(t)$. The speed is greatest wherever the slope of the velocity curve is zero, that is, wherever the acceleration is zero.

$$a(t) = -2\cos t - 3\sin t = 0$$

Solve for t.

$$\tan t = -\frac{2}{3}$$

$$t = \tan^{-1}\left(-\frac{2}{3}\right) + n\pi$$

The first five values of t that the speed is greatest are

$$t_1 \approx 2.55$$

$$t_2 \approx 5.70$$

$$t_3 \approx 8.84$$

$$t_4 \approx 11.98$$

$$t_5 \approx 15.12.$$